UDC 532.516

NONLINEAR PERTURBATION DEVELOPMENT IN TWO-DIMENSIONAL LAMINAR FLOWS

PMM Vol. 41, №5, 1977, pp. 802-806
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(Received October 20, 1975)

The problem with initial data for perturbations in laminar flows is considered. A modification of the method proposed in [1] is used for investigating the development of perturbations with time. The problem is solved for initial conditions and Reynolds numbers close to the critical value defined in the theory of stability. Conditions of secondary flow existence are obtained by the analysis of the expression for perturbations, and their stability is investigated. The results related to secondary flows are in agreement with those presented in [2, 3].

An approximate formula for the variation of perturbation amplitude with time was proposed by Landau [4]. His ideas were further developed in [5, 6]. A method similar to that of Poincaré was proposed in [1] for analyzing nonlinear equations for perturbations, and in [2, 3] the conditions of existence and the stability of self-oscillating modes that originate at the loss of laminar flow stability were investigated.

Let us consider the two-dimensional flow of a viscous incompressible fluid. The equations for the stream function $\psi(x, y, t)$ are of the form

$$\frac{\partial \Delta\psi}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} - \frac{1}{R}\Delta^2\psi = 0 \tag{1}$$

where R is the Reynolds number.

Substituting $\psi = \psi_0(x, y) + \Phi(x, y, t)$, where $\psi_0(x, y)$ is the stream function of the stationary laminar flow and $\Phi(x, y, t)$ is the perturbation stream function, into (1) for $\Phi(x, y, t)$ we obtain the equation

$$\frac{\partial\Delta\Phi}{\partial t} - M(\psi_0, \Phi) - \frac{1}{R}\Delta^2\Phi = N(\Phi, \Phi) \tag{2}$$

$$N(\psi, \varphi) = \frac{\partial\psi}{\partial x}\frac{\partial\Delta\varphi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\Delta\varphi}{\partial x}, \quad M(\psi, \varphi) = N(\psi, \varphi) + N(\varphi, \psi)$$

The form of boundary conditions for Φ depends on the particulars of a problem. Below we consider one of the following cases.

1°. The flow takes place in the bounded region Ω whose boundary is $\partial\Omega$, and the boundary conditions for Φ are

$$\Phi|_{\partial\Omega} = \frac{\partial\Phi}{\partial n}\Big|_{\partial\Omega} = 0$$

2°. The flow is periodic with respect to x $(\psi_0 \equiv \psi_0(y))$ with a bounded fundamental region of the three-dimensional periodicity $\Omega = \{x, y: 0 \leqslant x \leqslant T;\ y_1 \leqslant y \leqslant y_2\}$. The boundary conditions are

$$\Phi|_{y=y_1,y_2} = \frac{\partial \Phi}{\partial y}\bigg|_{y=y_1,\,y_2} = 0, \quad \Phi(x+T, y, t) = \Phi(x, y, t)$$

3°. The flow in the boundary layer is considered in a plane-parallel approximation. The three-dimensional periodicity region is $\Omega = \{x, y: 0 \leqslant x \leqslant T;\ 0 \leqslant y < \infty\}$, and the boundary conditions are

$$\Phi(x+T, y, t) = \Phi(x, y, t), \quad \Phi|_{y=0} = \frac{\partial \Phi}{\partial y}\bigg|_{y=0} = 0$$

$$\left|\frac{\partial \Phi_0}{\partial y}\right| \leqslant K \quad \text{for } y \to \infty; \quad \Phi_1, \ \frac{\partial \Phi_1}{\partial y} \to 0 \quad \text{for} \quad y \to \infty$$

$$\left(\Phi_0 = \frac{1}{T}\int_0^T \Phi\, dx, \quad \Phi_1 = \Phi - \Phi_0\right)$$

(It can be shown that in this case the results obtained in [2] are valid.)

We denote by W the set of functions that satisfy the boundary conditions of any of these problems, and seek the solution $\Phi \in W$ of the nonlinear equation (2) by using a method similar to that in [1]. We set

$$\Phi(x, y, t) = \sum_{n=1}^{\infty} A^n(\xi_1)\, \Phi_n(x, y, \xi_2) \tag{3}$$

$$\frac{d\xi_1}{dt} = \gamma + \sum_{n=1}^{\infty} b_n A^n(\xi_1), \quad \frac{d\xi_2}{dt} = \omega + \sum_{n=1}^{\infty} c_n A^n(\xi_1) \tag{4}$$

$$A(\xi_1) = \lambda e^{\xi_1}; \quad \xi_1 = 0, \quad \xi_2 = \theta \quad \text{for} \quad t = 0$$

where λ and θ are the initial amplitude and phase, respectively, and $\Phi_n(x, y, \xi_2)$ are periodic functions of ξ_2 of period 2π. We assume that λ is a small parameter. Substituting (3) and (4) into (2) and equating terms of like order of λ, for Φ_n we obtain the equations

$$n\gamma\Delta\Phi_n + \omega\frac{\partial \Delta\Phi_n}{\partial \xi_2} - L\Phi_n = -\sum_{l+k=n} b_l k \Delta\Phi_k - \sum_{l+k=n} c_l \frac{\partial \Delta\Phi_k}{\partial \xi_2} + \sum_{l+k=n} N(\Phi_l, \Phi_k), \quad \Phi_n \in W \tag{5}$$

$$L\varphi = M(\psi_0, \varphi) + R^{-1}\Delta^2\varphi$$

In what follows we assume that the linear boundary value problem

$$i\omega\Delta\varphi - L\varphi = 0, \quad \varphi \in W \tag{6}$$

has two simple real eigenvalues $\pm\,\omega_0$ when $R = R_0$ and that among the numbers $n\omega_0$ (n is an integer and $n \neq \pm 1$) there are no eigenvalues of problem (6). We seek the solution of system (5) in the form of series in the small parameter δ

$$\omega = \omega_0 + \sum_{k=1}^{\infty} \omega_k \delta^k, \quad \gamma = \sum_{k=1}^{\infty} \gamma_k \delta^k, \quad b_n = \sum_{k=0}^{\infty} b_{nk} \delta^k \tag{7}$$

$$c_n = \sum_{k=0}^{\infty} c_{nk} \delta^k, \quad \Phi_n = \sum_{k=0}^{\infty} \delta^k \Phi_{nk}(x, y, \xi_2), \quad \delta = \frac{1}{R_0} - \frac{1}{R}$$

Substituting (7) into (5) and equating terms of like powers of δ, we obtain

$$\omega_0 \frac{\partial \Delta \Phi_{nk}}{\partial \xi_2} - L_0 \Phi_{nk} = -\Delta^2 \Phi_{n,k-1} - n \sum_{i+j=k} \gamma_i \Delta \Phi_{nj} - \tag{8}$$

$$\sum_{i+j=k} \omega_i \frac{\partial \Delta \Phi_{nj}}{\partial \xi_2} - \sum_{i+j=k} \sum_{l+m=n} b_{li} m \Delta \Phi_{mj} -$$

$$\sum_{i+j=k} \sum_{l+m=n} c_{li} \frac{\partial \Delta \Phi_{mi}}{\partial \xi_2} + \sum_{i+j=k} \sum_{l+m=n} N(\Phi_{li}, \Phi_{mj})$$

$$\Phi_{nk} \in W \qquad (L_0 \varphi = L\varphi|_{R=R_0})$$

For Φ_{10} the right-hand side of Eq.(8) is zero. Taking into account that Φ_{10} is a periodic function of ξ_2 we obtain

$$\Phi_{10} = u(x, y) e^{i\xi_2} + u^*(x, y) e^{-i\xi_2} \tag{9}$$

where $u(x, y)$ is the eigenfunction of problem (6) when $R = R_0$ and $\omega = \omega_0$.

We introduce the normalization conditions

$$\int_\Omega v^*(x, y) \Delta u(x, y)\, dx\, dy = 1 \tag{10}$$

$$\int_0^{2\pi} \int_\Omega v^* e^{-i\xi_2} \Delta \Phi_{nk}\, dx\, dy\, d\xi_2 = 0 \quad \text{for} \quad n \neq 1,\ k \neq 0$$

where $v(x, y)$ is the solution of the conjugate equation of (6) in $L_2(\Omega)$ when $R = R_0$ and $\omega = \omega_0$.

The conditions of solvability of the nonhomogeneous equations for Φ_{nk} when $n \neq 1$ and $k \neq 0$, which are similar to those in [2], determine the unknown constants γ_k, ω_k, c_{nk}, and b_{nk}. Coefficients γ_k and ω_k are determined by the conditions of solvability of equations for Φ_{1k} $(k \geqslant 1)$. In particular

$$\gamma_1 = -\operatorname{Re} J, \quad J = \int_\Omega v^* \Delta^2 u\, dx\, dy \tag{11}$$

(formula (11) was obtained with allowance for the normalization condition (10)). Functions Φ_{1k} $(k = 1, 2, \ldots)$ are of the form

$$\Phi_{1k} = u_k(x, y) e^{i\xi_2} + u_k^*(x, y) e^{-i\xi_2} \tag{12}$$

Let us consider the system of Eqs. (8) when $n = 2$. The conditions of solvability with allowance for (9) and (12) yield

$$b_{1k} = c_{1k} = 0, \quad k = 0, 1, 2 \ldots$$

The normalization condition (10) uniquely determines Φ_{2k}

$$\Phi_{2k} = \varphi_{k0}(x, y) + \varphi_{k2}(x, y)e^{2i\xi_2} + \varphi_{k2}^*(x, y)e^{-2i\xi_2}, \quad k = 0, 1, \ldots$$

Coefficients b_{2k} and c_{2k} are determined by the conditions of solvability of system (8) when $n = 3$. In particular

$$b_{20} = \operatorname{Re} I, \quad c_{20} = \operatorname{Im} I \tag{13}$$

$$I = \int_0^{2\pi}\int_\Omega e^{-i\xi_2}v^* M(\Phi_{10}, \Phi_{20})\, dx\, dy\, d\xi_2$$

By successively solving system (8) we can determine all functions Φ_{nk} and coefficients b_{nk} and c_{nk}. Using the method of induction it is possible to show that Φ_{nk} $(k = 0, 1, 2 \ldots)$ for even n contains only even harmonics and for odd n the harmonics are odd, and that $b_{nk} = c_{nk} = 0$ $(k = 0, 1, 2 \ldots)$ when n is odd.

Let us consider Eq. (4) for ξ_1.

Let γ_1 and b_{20} be nonzero. Then in the first approximation with respect to δ and λ^2 we obtain

$$t = \int_0^{\xi_1} \frac{d\xi_1}{\gamma_1\delta + b_{20}A^2} \tag{14}$$

(If γ_1 or b_{20} are zero, it is necessary to take into account in formula (4) the next terms of expansion in δ and λ^2.) Substituting $A = \lambda e^{\xi_1}$ we obtain for $A(t)$ the expression

$$A(t) = \left(\frac{d\lambda^2 e^{2\gamma_1\delta t}}{d - \lambda^2 + \lambda^2 e^{2\gamma_1 t}}\right)^{1/2}, \quad d = -\frac{\gamma_1\delta}{b_{20}} \tag{15}$$

Noting that γ is the coefficient of perturbation increase in the linear theory of stability, we shall investigate formula (15).

Let $\gamma_1\delta > 0$ (the supercritical case).

If $d > 0$, then $A(t) \to d^{1/2} \equiv A_0$ when $t \to \infty$, i.e. there exists a stable secondary flow. From (11) and (13) we obtain

$$A_0 = \left(\frac{\operatorname{Re} J}{\operatorname{Re} I}\,\delta\right)^{1/2} \tag{16}$$

When $d < 0$, then $A \to \infty$ when $t \to (2\gamma_1\delta)^{-1}\ln(1 + |d| / \lambda^2)$, i.e. there are no equilibrium modes of small amplitude.

Let $\gamma_1\delta < 0$ (the subcritical case).

If $d > 0$, a secondary equilibrium mode obtains for the unique initial value $\lambda_0 = d^{1/2} \equiv A_0$. When $\lambda < \lambda_0$, perturbations are attenuated, while for $\lambda > \lambda_0$ they infinitely increase in the finite time interval $t = (2\gamma_1\delta)^{-1}\ln(1 - d / \lambda^2)$, i.e. the secondary subcritical modes are unstable.

When $d < 0$ all perturbations are attenuated, and there are no secondary flows of small amplitude.

Let $\delta = 0$ (the indifferent case).

From (15) we have

$$A(t) = \lambda(1 - 2\lambda^2 b_{20} t)^{-1/2}$$

When $b_{20} < 0$, the perturbations attenuate, while for $b_{20} > 0$ they infinitely increase in the finite time interval $t = (2\lambda^2 b_{20})^{-1}$.

Formula (16) for the amplitude, the region of existence, and the form of stream functions of secondary equilibrium flows conform to the results presented in [2]. It can be shown that the derived series (3) and (7) are asymptotic expansions of the solution of Eq.(2) in small parameters δ and λ.

The quantity γ_1/b_{20} was numerically determined for some simple flows in several papers in connection with the investigation of secondary equilibrium modes. For example, for a plane Poiseuille flow, in (7) and for a Blasius flow, in [8].

When $\mathrm{Re}\, J \neq 0$, γ may be taken as the parameter of the expansion for solving system (5). Applying a reasoning similar to the above, it is possible to show that in the first approximation with respect to γ and λ^2 the formula for $A(t)$ is of the form (15) in which $\gamma_1\delta = \gamma$ and $d = -\gamma/b_{20}$. The relation between δ and γ is defined by formula

$$\delta = \sum_{k=1}^{\infty} \delta_k \gamma^k, \quad \delta_1 = -\frac{1}{\mathrm{Re}\, J}$$

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Translated by J.J. D.